Problem of the Week
Grade 7 and 8

Fractions to the Max
Solution

Problem

In the expression $\frac{a}{b} + \frac{c}{d} + \frac{e}{f}$, each letter is replaced by a different digit from 1,2,3,4,5,6. Each digit can be used exactly once. What is the largest possible value of this expression?

Solution 1

The largest fractions will be created by putting the three smallest numbers, 1, 2, and 3, in the denominators and then placing the numbers 4, 5, and 6 in the numerators. We will do this in every possible way, determine the sums and choose the largest.

There are six different possible sums in which 1, 2, and 3 are the denominators and 4, 5, 6 are the numerators.

$$
\begin{array}{ll}
(1) & \frac{4}{1} + \frac{5}{2} + \frac{6}{3} = \frac{24}{6} + \frac{15}{6} + \frac{12}{6} = \frac{51}{6} \\
(2) & \frac{4}{1} + \frac{6}{2} + \frac{5}{3} = \frac{24}{6} + \frac{18}{6} + \frac{10}{6} = \frac{52}{6} \\
(3) & \frac{5}{1} + \frac{4}{2} + \frac{6}{3} = \frac{30}{6} + \frac{12}{6} + \frac{12}{6} = \frac{54}{6} \\
(4) & \frac{5}{1} + \frac{6}{2} + \frac{4}{3} = \frac{30}{6} + \frac{18}{6} + \frac{8}{6} = \frac{56}{6} \\
(5) & \frac{6}{1} + \frac{4}{2} + \frac{5}{3} = \frac{36}{6} + \frac{12}{6} + \frac{10}{6} = \frac{58}{6} \\
(6) & \frac{6}{1} + \frac{5}{2} + \frac{4}{3} = \frac{36}{6} + \frac{15}{6} + \frac{8}{6} = \frac{59}{6}
\end{array}
$$

Therefore the largest possible value of the expression is $\frac{59}{6}$. It should be noted that this approach would not be practical if more numbers were involved. Be sure to look at solution 2 for a more logical approach.

Solution 2

We can start by observing that to get a fraction with the highest value we need a 6 in the numerator. The choice of denominators is possibly obvious as well. $\frac{6}{1} = 6$, $\frac{6}{2} = 3$, $\frac{6}{3} = 2$, $\frac{6}{4} = 1.5$, and $\frac{6}{5} = 1.2$. $\frac{6}{1}$ is the largest fraction and any numerator other than 6 will produce a lower value.

Now we have four numbers left to place: $\{2,3,4,5\}$.

Of these remaining numbers, since 5 is the largest it should go in the numerator. Then $\frac{5}{2} = 2.5$, $\frac{5}{3} \doteq 1.7$, and $\frac{5}{4} = 1.25$. $\frac{5}{2}$ is the largest fraction and any numerator other than 5 will produce a lower value.

Now we have two numbers left to place: $\{3,4\}$.

Our only two choices for the third fraction are $\frac{4}{3} \doteq 1.3$, and $\frac{3}{4} = 0.75$. So $\frac{4}{3}$ is the third fraction.

We can now determine the largest possible sum.

$$\begin{aligned}\text{Largest Possible Sum} &= \frac{6}{1} + \frac{5}{2} + \frac{4}{3} \\ &= \frac{36}{6} + \frac{15}{6} + \frac{8}{6} \\ &= \frac{59}{6} \text{ or } 9\frac{5}{6}\end{aligned}$$

$\therefore$ the largest possible sum that can be made from the numbers 1, 2, 3, 4, 5, and 6 in the expression $\dfrac{a}{b} + \dfrac{c}{d} + \dfrac{e}{f}$ is $\frac{59}{6}$ or $9\frac{5}{6}$ (approximately 9.83).